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QUANTUM ORTHOGONAL CAYLEY-KLEIN GROUPS IN CARTESIAN BASIS

N.A. GROMOV, I.V. KOSTYAKOV and V.V. KURATOV

Department of Mathematics, Komi Science Centre,
Ural Division, Russian Academy of Sciences,
Syktyvkar, 167000, Russia
E-mail : parma@omkomi.intec.ru

Abstract

The similarity transformations of quantum orthogonal groups are developed and FRT theory is reformulated to the Cartesian basis. The quantum orthogonal Cayley-Klein groups are introduced as the algebra functions over an associative algebra with the nilpotent generators. The quantum orthogonal Cayley-Klein algebras are obtained as the dual objects to the corresponding quantum groups.

1 Introduction

The *simple* quantum groups and algebras are described by FRT theory[1]. Although the quantum group as a mathematical structure is not a group at all but it is a Hopf algebra, nevertheless there are many parallel features for both group and quantum group.

In group theory there is a remarkable set of groups, namely the motion groups of 3^n n-dimensional spaces of constant curvature or the orthogonal Cayley-Klein (CK) groups[2]. The classical simple orthogonal groups $SO(n)$, the semisimple pseudoorthogonal groups $SO(p,q)$ and nonsemisimple groups such as Euclidean $E(n)$, Poincare $P(n)$, Galileian $G(n)$ are in this set. The nonsemisimple groups may be obtained from the simple (or semisimple) one's by the well known procedure of contraction[3]. On the other hand the set of CK groups may be described in uniform manner with the help of pure algebraical methods[2].

In the present paper we apply this approach for description of quantum (nonsemisimple) orthogonal CK groups and corresponding quantum algebras. As it turned out slightly modified FRT theory is suitable for this purpose: it is enough to replace the complex number field $\mathbf{C}$ by the dual algebra $\mathbf{D}_n(\iota;\mathbf{C})$ with the nilpotent generators.

The contents of this work is as follows. The dual algebra $\mathbf{D}_n(\iota;\mathbf{C})$ is briefly described in Sec.2. The linear transformations of generators of

quantum orthogonal group are regarded in Sec.3 and the description of $SO_q(N;\mathbf{C})$ is given. In Sec.4 the quantum Cayley-Klein groups $SO_v(N;j)$ are introduced as Hopf algebra of the noncommutative functions with dual variables. Quantum Cayley-Klein algebra $so_v(N;j)$ are cosidered in Sec.5 as the dual object to the $SO_v(N;j)$. The developed theory is illustrated by the N=3 example of quantum group and algebra in Sec.6.

2 Dual algebra $\mathbf{D}_n(\iota;\mathbf{C})$

Dual algebra $\mathbf{D}_n(\iota;\mathbf{C})$ is defined as an associative algebra with unit and *nilpotent* generators $\iota_1,\ldots,\iota_n,\ \iota_k^2=0,\ k=1,\ldots,n$ with *commutative* multiplication $\iota_k\iota_m=\iota_m\iota_k,\ k\neq m$. The general element of $\mathbf{D}_n(\iota;\mathbf{C})$ has the form

$$a=a_0+\sum_{p=1}^{2^n-1}\sum_{k_1<\ldots<k_p}a_{k_1\ldots k_p}\iota_{k_1}\ldots\iota_{k_p},\quad a_0,a_{k_1\ldots k_p}\in\mathbf{C}. \tag{1}$$

For $n=1$ we have $\mathbf{D}_1(\iota_1;\mathbf{C})\ni a=a_0+a_1\iota_1$, i.e. dual (or Study) numbers, when $a_0,a_1\in\mathbf{R}$. For $n=2$ the general element of $\mathbf{D}_2(\iota_1,\iota_2;\mathbf{C})$ is written as follows: $a=a_0+a_1\iota_1+a_2\iota_2+a_{12}\iota_1\iota_2$.

Divisions $a/\iota_k, a\in\mathbf{R},\mathbf{C}$, and $\iota_m/\iota_k, k\neq m$ are not defined, but division of a dual unit by itself is equal to real unit $\iota_k/\iota_k=1$. A function of a dual argument is defined by its Taylor expansion $f(x_0+\iota_k x_1)=f(x_0)+\iota_k x_1 f^{'}(x_0)$.

The well known Grassmann algebra $\Gamma_n(\xi)$ is the algebra with *nilpotent* generators $\xi_k^2=0, k=1,\ldots,n$ and *anticommutative* multiplication $\xi_k\xi_m=-\xi_m\xi_k, k\neq m$. It is easy to verify that the product of two generators of Grassmann algebra has the same algebraic properties as the generator of dual algebra $\iota_k=\xi_k\xi_{n+k},\ k=1,\ldots,n$. This means that the dual algebra is the subalgebra of even part of Grassmann algebra $\mathbf{D}_n(\iota;\mathbf{C})\subset\Gamma_{2n}(\xi)$.

3 Cartesian basis for $SO_q(N;\mathbf{C})$

According with FRT theory[1] of quantum groups let us regard an algebra $\mathbf{C}\langle t_{ik}\rangle$ of noncommutative polynomials of N^2 variables $t_{ik}, i,k=1,\ldots,N$ over complex number field $\mathbf{C}$. For well known[1] lower triangular matrix $R_q\in M_{N^2}(\mathbf{C})$ the generators $T=(t_{ik})_{i,k=1}^N\in M_N(\mathbf{C}\langle t_{ik}\rangle)$ have the following commutation relations

$$RT_1T_2=T_2T_1R, \tag{2}$$

where $T_1=T\otimes I,\ T_2=I\otimes T\in M_{N^2}(\mathbf{C}\langle t_{ij}\rangle)$. There are additional relations of q-orthogonality

$$TCT^t=T^tCT=C, \tag{3}$$

where $C=C_0q^\rho, \rho=diag(\rho_1,\ldots,\rho_N)$,

$$(\rho_1,\ldots,\rho_N)=\begin{cases}(n-\frac{1}{2},n-\frac{3}{2},\ldots,\frac{1}{2},0,-\frac{1}{2},\ldots,-n+\frac{1}{2}), & \text{for}\quad N=2n+1\\ (n-1,n-2,\ldots,1,0,0,-1,\ldots,-n+1), & \text{for}\quad N=2n.\end{cases} \tag{4}$$

and the only nonzero elements of matrix $C_0 \in M_N(\mathbf{C})$ are real units on the second diagonal $(C_0)_{ik} = \delta_{i'k}, i' = N+1-i$. The quantum orthogonal group $SO_q(N;\mathbf{C})$ is defined as the quotient

$$SO_q(N;\mathbf{C}) = \mathbf{C}\langle t_{ik}\rangle \big/ (2),(3). \tag{5}$$

¿From the algebraic point of view $SO_q(N;\mathbf{C})$ is a Hopf algebra with the following coproduct Δ, counit ϵ and antipode S :

$$\Delta T = T\dot{\otimes}T, \quad \epsilon(T) = I, \quad S(T) = CT^tC^{-1}. \tag{6}$$

For $q = 1$ Eq.(3) is the orthogonality condition written with the help of matrix C_0 and the set of matrices T is the orthogonal group $SO(N;\mathbf{C})$ in so-called "symplectic" basis. We shall call $SO_q(N;\mathbf{C})$ described by Eqs.(2)–(5) as the orthogonal quantum group in symplectic basis.

One of the solutions of matrix equation

$$DC_0D^t = I, \tag{7}$$

have the form

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I & \tilde{C}_0 \\ i\tilde{C}_0 & -iI \end{pmatrix}, \; N = 2n,$$

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 & \tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ i\tilde{C}_0 & 0 & -iI \end{pmatrix}, \; N = 2n+1, \tag{8}$$

where $\tilde{C}_0 \in M_n(\mathbf{C})$ is the matrix with the real units on the second diagonal. For the nondeformed case $q = 1$ the equation

$$U = DTD^{-1} \tag{9}$$

define the similarity transformation of the orthogonal matrix from symplectic to Cartesian basis. So we shall call the linear transformation (9) of generators of quantum group $SO_q(N;\mathbf{C})$ as transformation to the Cartesian basis.

It is easy to reformulate Eqs.(2)–(5) to the new basis. Commutation relations of Cartesian (or rotation) generators $U = (u_{ik})_{i,k=1}^N \in M_N(\mathbf{C}\langle u_{ik}\rangle)$ are given by

$$\tilde{R}_qU_1U_2 = U_2U_1\tilde{R}_q, \tag{10}$$

where

$$\tilde{R}_q = (D\otimes D)R_q(D\otimes D)^{-1} \tag{11}$$

is now the nontriangular matrix. The additional relations of q-orthogonality are as follows

$$UC'U^t = C', \quad U^t(C')^{-1}U = (C')^{-1}, \tag{12}$$

where

$$C' = DCD^t == \begin{pmatrix} \cosh z\tilde{\rho} & 0 & i\sinh z\tilde{\rho}_0 \\ 0 & 1 & 0 \\ -i\tilde{C}_0\sinh z\tilde{\rho} & 0 & \tilde{C}_0\cosh z\tilde{\rho}_0 \end{pmatrix}, \tag{13}$$

$\tilde{\rho} = diag(\rho_1,\ldots,\rho_n) \in M_n(\mathbf{C})$, for $N = 2n+1$ and without middle column and row for $N = 2n$.

The orthogonal quantum group $SO_q(N;\mathbf{C})$ in Cartesian basis is described by the quotient

$$SO_q(N;\mathbf{C}) = \mathbf{C}\langle u_{ik}\rangle \big/ (10),(12) \tag{14}$$

and is Hopf algebra with the following coproduct Δ, counit ϵ and antipode S :

$$\Delta U = U \dot{\otimes} U, \quad \epsilon(U) = I, \quad S(U) = C' U^t (C')^{-1}. \tag{15}$$

4 Quantum CK groups $SO_v(N;j)$ as matrix groups over dual algebras $\mathbf{D}_{N-1}(\iota)$.

Orthogonal Cayley-Klein (CK) groups $SO(N;j)$ (or motion groups of spaces of constant curvature)[2] are realized in the Cartesian basis as the matrix groups over $\mathbf{D}_{N-1}(\iota)$ with the help of the *special* matrices

$$(A(j))_{kp} = \tilde{J}_{kp} a_{kp}, \quad a_{kp} \in C,$$

$$\tilde{J}_{kp} = J_{kp},\ k < p, \quad \tilde{J}_{kp} = J_{pk},\ k \geq p, \quad J_{\mu\nu} = \prod_{r=\mu}^{\nu-1} j_r,\ \mu < \nu,\ j_r = 1, \iota_r, i. \tag{16}$$

These matrices are satisfied the following j-orthogonality relations:

$$A(j)A^t(j) = A^t(j)A(j) = I. \tag{17}$$

In the symplectic basis the orthogonal CK groups are described by the matrices

$$B(j) = D^{-1}A(j)D \tag{18}$$

with the additional relations of j-orthogonality

$$B(j)C_0B^t(j) = B^t(j)C_0B(j) = C_0. \tag{19}$$

The matrix D in Eqs.(18) is given by Eqs.(8).

For example matrices from CK group $SO(3;j)$ in Cartesian basis are

$$A(j) = \begin{pmatrix} a_{11} & j_1 a_{12} & j_1 j_2 a_{13} \\ j_1 a_{21} & a_{22} & j_2 a_{23} \\ j_1 j_2 a_{31} & j_2 a_{32} & a_{33} \end{pmatrix} \tag{20}$$

and in symplectic basis are

$$B(j) = \begin{pmatrix} b_{11} + ij_1j_2\tilde{b}_{11} & j_1 b_{12} - ij_2\tilde{b}_{12} & b_{13} - ij_1j_2\tilde{b}_{13} \\ j_1 b_{21} + ij_2\tilde{b}_{21} & b_{22} & j_1 b_{21} - ij_2\tilde{b}_{21} \\ b_{13} + ij_1j_2\tilde{b}_{13} & j_1 b_{12} + ij_2\tilde{b}_{12} & b_{11} - ij_1j_2\tilde{b}_{11} \end{pmatrix}. \tag{21}$$

We shall regard the quantum deformations of the contracted CK groups, i.e. $j_k = 1, \iota_k$. We shall follow to FRT theory[1], but instead of the algebra $\mathbf{C}\langle t_{ik}\rangle$ our starting point is $\mathbf{D}\langle t_{ik}\rangle$ — the algebra of noncommutative polynomials of N^2 variables $t_{ik},\ i,k = 1,\ldots,N$ over dual algebra $\mathbf{D}_{N-1}(\iota)$. In addition we shall transform the deformation parameter $q = \exp z$ as follows:

$$z = Jv, \quad J \equiv J_{1N} = \prod_{k=1}^{N-1} j_k, \tag{22}$$

where v is the new deformation parameter.

In symplectic basis the quantum CK group $SO_v(N;j)$ is produced by the generating matrix $T(j) \in M_N(\mathbf{D}\langle t_{ik}\rangle)$ equal to B(j) (18) for $q=1$. The noncommutative entries of $T(j)$ obey the commutation relations

$$R_v(j)T_1(j)T_2(j) = T_2(j)T_1(j)R_v(j). \tag{23}$$

and the additional relations of (v,j)-orthogonality

$$T(j)C(j)T^t(j) = T^t(j)C(j)T(j) = C(j), \tag{24}$$

where lower triangular R–matrix $R_v(j)$ and $C(j)$ are obtained from R_q and C, respectively, by substitution Jv instead of $z:$

$$R_v(j) = R_q(z \to Jv), \quad C(j) = C(z \to Jv). \tag{25}$$

Then the quotient

$$SO_v(N;j) = \mathbf{D}\langle t_{ik}\rangle \big/ (23),(24) \tag{26}$$

is Hopf algebra with the coproduct Δ, counit ϵ and antipode $S:$

$$\Delta T(j) = T(j)\dot{\otimes}T(j), \quad \epsilon(T(j)) = I, \quad S(T(j)) = C(j)T^t(j)C^{-1}(j). \tag{27}$$

In Cartesian basis the quantum CK group $SO_v(N;j)$ is generated by $U(j) = (\tilde{J}_{ik}u_{ik}) \in M_N(\mathbf{D}\langle u_{ik}\rangle)$ with the commutation relations

$$\tilde{R}_v(j)U_1(j)U_2(j) = U_2(j)U_1(j)\tilde{R}_v(j) \tag{28}$$

and additional relations of (v,j)-orthogonality

$$U(j)C'(j)U^t(j) = C'(j), \quad U^t(j)(C'(j))^{-1}U(j) = (C'(j))^{-1}, \tag{29}$$

where

$$\tilde{R}_v(j) = \tilde{R}_q(z \to Jv), \quad C'(j) = C'(z \to Jv). \tag{30}$$

The quotient

$$SO_v(N;j) = \mathbf{D}\langle u_{ik}\rangle \big/ (28),(29) \tag{31}$$

is Hopf algebra with the following coproduct Δ, counit ϵ and antipode $S:$

$$\Delta U(j) = U(j)\dot{\otimes}U(j), \quad \epsilon(U(j)) = I, \quad S(U(j)) = C^{'}(j)U^t(j)(C^{'}(j))^{-1}. \tag{32}$$

5 Quantum CK algebras $so_v(N;j)$ as a dual to $SO_v(N;j)$.

According to FRT theory[1] the dual space $Hom(SO_v(N;j),\mathbf{C})$ is an algebra with the multiplication induced by coproduct Δ in $SO_v(N;j)$

$$l_1 l_2(a) = (l_1 \otimes l_2)(\Delta(a)), \tag{33}$$

$l_1, l_2 \in Hom(SO_v(N;j),\mathbf{C}), \quad a \in SO_v(N;j)$. Let us formally introduce $N \times N$ upper $(+)$ and lower $(-)$ triangular matrices $L^{(\pm)}(j)$ as follows: it is necessary to put j_k^{-1} in the nondiagonal matrix elements of $L^{(\pm)}(j)$, if

there is the parameter j_k in the corresponding matrix element of $T(j)$. For example, if $(T(j))_{12} = j_1 t_{12} + j_2 \tilde{t}_{12}$, then $(L^{(+)}(j))_{12} = j_1^{-1} l_{12} + j_2^{-1} \tilde{l}_{12}$. Formally the matrices $L^{(\pm)}(j)$ are not defined for $j_k = \iota_k$, since ι_k^{-1} do not exist, but if we give an action of the matrix functionals $L^{(\pm)}(j)$ on the elements of $SO_v(N;j)$ by the duality relation

$$\langle L^{(\pm)}(j), T(j) \rangle = R^{(\pm)}(j), \tag{34}$$

where

$$R^{(+)}(j) = P R_v(j) P, \quad R^{(-)}(j) = R_v^{-1}(j), \quad P u \otimes w = w \otimes u, \tag{35}$$

then we shall have well defined expressions even for dual values of the parameters j_k.

The elements of $L^{(\pm)}(j)$ satisfy the commutation relations

$$\begin{aligned} R^{(+)}(j) L_1^{(\sigma)}(j) L_2^{(\sigma)}(j) &= L_2^{(\sigma)}(j) L_1^{(\sigma)}(j) R^{(+)}(j), \\ R^{(+)}(j) L_1^{(+)}(j) L_2^{(-)}(j) &= L_2^{(-)}(j) L_1^{(+)}(j) R^{(+)}(j), \quad \sigma = \pm \end{aligned} \tag{36}$$

and additional relations

$$\begin{aligned} L^{(\pm)}(j) C^t(j) L^{(\pm)}(j) &= C^t(j), \\ L^{(\pm)}(j) (C^t(j))^{-1} L^{(\pm)}(j) &= (C^t(j))^{-1}, \\ l_{kk}^{(+)} l_{kk}^{(-)} = l_{kk}^{(-)} l_{kk}^{(+)} = 1, & \quad l_{11}^{(+)} \ldots l_{NN}^{(+)} = 1, \ k = 1, \ldots, N. \end{aligned} \tag{37}$$

An algebra $so_v(N;j) = \{I, L^{(\pm)}(j)\}$ is called quantum CK algebra and is Hopf algebra with the following coproduct Δ, counit ϵ and antipode $S:$

$$\Delta L^{(\pm)}(j) = L^{(\pm)}(j) \dot{\otimes} L^{(\pm)}(j), \quad \epsilon(L^{(\pm)}(j)) = I,$$
$$S(L^{(\pm)}(j)) = C^t(j) (L^{(\pm)}(j))^t (C^t(j))^{-1}. \tag{38}$$

It is possible to show that algebra $so_v(N;j)$ is isomorphic with the quantum deformation of the universal enveloping algebra of the CK algebra $so(N;j)$, which may be obtained from the orthogonal algebra $so(N)$ by contractions[2]. So there are at least two ways for constraction of quantum CK algebras.

6 Example: $SO_v(3;j)$ and $so_v(3;j)$ in symplectic basis

The generating matrix for the simplest quantum orthogonal group $SO_v(3;j), \ j = (j_1, j_2)$ is in the form

$$T(j) = \begin{pmatrix} t_{11} + i j_1 j_2 \tilde{t}_{11} & j_1 t_{12} - i j_2 \tilde{t}_{12} & t_{13} - i j_1 j_2 \tilde{t}_{13} \\ j_1 t_{21} + i j_2 \tilde{t}_{21} & t_{22} & j_1 t_{21} - i j_2 \tilde{t}_{21} \\ t_{13} + i j_1 j_2 \tilde{t}_{13} & j_1 t_{12} + i j_2 \tilde{t}_{12} & t_{11} - i j_1 j_2 \tilde{t}_{11} \end{pmatrix}. \tag{39}$$

The R-matrix is obtained from the standart one by Eq.(25) and is as follows

$$R_v(j) \equiv R_q(z \to Jv) =$$

$$
\begin{pmatrix}
e^{Jv} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-Jv} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\sinh Jv & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2e^{-Jv/2}\sinh Jv & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2(1-e^{-Jv})\sinh Jv & 0 & -2e^{-Jv/2}\sinh Jv & 0 & e^{-Jv} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\sinh Jv & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{Jv}
\end{pmatrix}. \tag{40}
$$

Qver the dual algebras $\mathbf{D}_2(j_1,j_2),\ j_1=\iota_1, j_2=1$, or $j_1=1, j_2=\iota_2$, or $j_1=\iota_1, j_2=\iota_2$ this R-matrix may be written in the form

$$
R_v(j) = I + Jv\tilde{R}, \tag{41}
$$

where

$$
\tilde{R} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{42}
$$

The commutation relations and additional relations of (v,j)-orthogonality may be obtained from Eqs.(23),(24) by straitforward calculations, so we shall concentrate our attention on the constraction of quantum algebra $so_v(3;j)$.

The matrix functionals $L^{\pm}(j)$ have the form

$$
L^{(+)}(j) = \begin{pmatrix}
l_{11} & j_1^{-1}l_{12} - ij_2^{-1}\tilde{l}_{12} & l_{13} - ij_1^{-1}j_2^{-1}\tilde{l}_{13} \\
0 & 1 & j_1^{-1}l_{21} - ij_2^{-1}\tilde{l}_{21} \\
0 & 0 & l_{11}^{-1}
\end{pmatrix}, \tag{43}
$$

$$
L^{(-)}(j) = \begin{pmatrix}
l_{11}^{-1} & 0 & 0 \\
j_1^{-1}l_{21} + ij_2^{-1}\tilde{l}_{21} & 1 & 0 \\
l_{13} + ij_1^{-1}j_2^{-1}\tilde{l}_{13} & j_1^{-1}l_{12} + ij_2^{-1}\tilde{l}_{12} & l_{11}
\end{pmatrix}. \tag{44}
$$

Their actions on the generators (39) of quantum group $SO_v(3;j)$ are given by Eq.(34) and are as follows:

$$
l_{11}(t_{22}) = 1, \quad l_{11}(t_{11}) = \cosh Jv, \quad \tilde{l}_{11}(\tilde{t}_{11}) = -J^{-1}\sinh Jv,
$$

$$
l_{12}(\tilde{t}_{21}) = -ij_1^2J^{-1}\sinh Jv, \quad l_{12}(\tilde{t}_{12}) = ij_1^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2),
$$

$$
l_{12}(t_{12}) = (\cosh 3Jv/2 - \cosh Jv/2)/2 = \tilde{l}_{12}(\tilde{t}_{12}), \quad \tilde{l}_{12}(t_{21}) = ij_2^2J^{-1}\sinh Jv,
$$

$$
\tilde{l}_{12}(t_{12}) = -ij_2^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2), \quad l_{21}(\tilde{t}_{12}) = -ij_1^2J^{-1}\sinh Jv,
$$

$$
l_{21}(\tilde{t}_{21}) = ij_1^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2), \quad \tilde{l}_{21}(t_{12}) = ij_2^2J^{-1}\sinh Jv,
$$

$$
\tilde{l}_{21}(t_{21}) = -ij_2^2(2J)^{-1}(\sinh 3Jv/2 + \sinh Jv/2), \quad l_{13}(t_{13}) = (\cosh 2Jv - 1)/2 = \tilde{l}_{13}(\tilde{t}_{13}),
$$

$$
l_{21}(t_{21}) = (\cosh 3Jv/2 - \cosh Jv/2)/2 = \tilde{l}_{21}(\tilde{t}_{21}),
$$

$$l_{13}(\tilde{t}_{13}) = -iJ^{-1}(2\sinh Jv - \sinh 2Jv), \quad \tilde{l}_{13}(t_{13}) = iJ(2\sinh Jv - \sinh 2Jv), \tag{45}$$

where $J = j_1 j_2$. Only nonzero expressions are written out above. According with the additional relations (37) there are three independent generators of $so_v(3;j)$, for example, $l_{11}, l_{12}, \tilde{l}_{12}$. Their commutation relations follow from Eq.(36)

$$l_{11} l_{12} \cosh Jz - l_{12} l_{11} = l_{11} \tilde{l}_{12} i j_1^2 J^{-1} \sinh Jz,$$
$$l_{11} \tilde{l}_{12} \cosh Jz - \tilde{l}_{12} l_{11} = -l_{11} l_{12} i j_2^2 J^{-1} \sinh Jz,$$
$$\left[l_{12}, \tilde{l}_{12}\right] = -\left(l_{11}^2 - 1\right) iJ \sinh Jz. \tag{46}$$

The quantum analogue of the universal enveloping algebra of CK algebra $so(3;j) = \{X_{01}, X_{02}, X_{12}\}$ with the rotation generator X_{02} as the primitive element of the Hopf algebra has been given in[4]. The Hopf algebra structure of $so_v(3;j;X_{02})$ is given by

$$\Delta X_{02} = I \otimes X_{02} + X_{02} \otimes I,$$
$$\Delta X = e^{-vX_{02}/2} \otimes X + X \otimes e^{vX_{02}/2}, \quad X = X_{01}, X_{12},$$
$$\epsilon(X_{01}) = \epsilon(X_{02}) = \epsilon(X_{12}) = 0, \quad S(X_{02}) = -X_{02},$$
$$S(X_{01}) = -X_{01} \cos Jv/2 + j_1^2 X_{12} J^{-1} \sin Jv/2,$$
$$S(X_{12}) = -X_{12} \cos Jv/2 - j_2^2 X_{01} J^{-1} \sin Jv/2,$$
$$[X_{01}, X_{02}] = j_1^2 X_{12}, \quad [X_{02}, X_{12}] = j_2^2 X_{01}, \quad [X_{12}, X_{01}] = \frac{\sinh vX_{02}}{v}. \tag{47}$$

The isomorphism of $so_v(3;j;X_{02})$ and quantum algebra $so_v(3;j)$ is easily established with the help of the following relations between generators

$$l_{11} = e^{vX_{02}}, \quad j_1^{-1} l_{12} = j_2 E X_{01} e^{vX_{02}/2}, \quad j_2^{-1} \tilde{l}_{12} = -j_1 E X_{12} e^{vX_{12}/2},$$
$$E = i\left(vJ^{-1} \sin Jv\right)^{1/2} e^{-Jv}. \tag{48}$$

So the quantum orthogonal CK algebras may be constracted both as the dual to the quantum group and by the contractions of quantum orthogonal algebras.

7 Conclusion

Contractions are the method of receiving a new Lie groups (algebras) from the unitial ones, in particular, the nonsemisimple qroups (algebras) from the simple ones. In the traditional approach[3] this is achieved by introduction of a real zero tending parameter $\epsilon \to 0$. In our approach[2], contractions are described by the dual valued parameters j_k. In the case of FRT theory of quantum groups these contractions supplemented with the appropriate transformations of the deformation parameter lead to realization of nonsemisimple quantum qroups as Hopf algebras of noncommutative functions with dual variables. So at least in CK scheme from mathematical point of view the contraction procedure is nothing else as the replacement of complex number field $\mathbf{C}$ by the algebra $\mathbf{D}(\iota)$ for both classical and guantum groups.

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